April 5, 2017

Intersection of Lines in 2D and 3D

In 2D, there are three cases for intersections:

a) one intersection (slopes are different)

b) infinite solutions (sharing a point and direction vector)

c) no solution (same direction vector but no common point)

Ex. 1: Find the solution for the following

$L_1: (2,3) + s[-4, 1]$

$L_2: (0,-1) + t[2,-3]$

$x = 2 - 4s \quad ; \quad x = 2t$

$y = 3 + s \quad ; \quad y = -1 - 3t$

set $x = x$, $y = y$

$2 - 4s = 2t$ ①

$3 + s = -1 - 3t$ ②

$$\boxed{t = 1 - 2s} \quad (3)$$

sub into (2)

$$3 + s = -1 - 3(1 - 2s)$$

$$3 + s = -1 - 3 + 6s$$

$$s = -7 + 6s$$

$$-5s = -7$$

$$\boxed{s = 7/5}$$

sub into (3)

$$t = 1 - 2\left(\frac{7}{5}\right)$$

$$\boxed{t = -\frac{9}{5}}$$

$x = 2 - 4s \quad ; \quad x = 2t$

$y = 3 + s \quad ; \quad y = -1 - 3t$

$x = 2 - 4\left(\frac{7}{5}\right) = 2 - \frac{28}{5} = \boxed{-\frac{18}{5}}$

$y = 3 + \left(\frac{7}{5}\right) = \boxed{\frac{22}{5}}$

other point

$x = 2\left(-\frac{9}{5}\right) = \boxed{-\frac{18}{5}}$

$y = -1 - 3\left(-\frac{9}{5}\right) = \boxed{\frac{22}{5}}$

$\therefore$ POI is $\left(-\frac{18}{5}, \frac{22}{5}\right)$

Intersection of 2 Lines in 3D

Here we have four cases

1. lines have different direction vectors and meet at a point (1 solution)
2. infinite solutions (sharing a point and direction vectors)
3. parallel lines for no solution (same direction vectors and no shared points).
4. skew lines (different direction vectors but lines don't intersect)

Ex. 2: Find the POI (if they intersect).

$L_1: (5,8,13) + s[2,1,3]$

$L_2: (3,2,2) + t[4,-3,-2]$

$x = 5 + 2s \qquad x = 3 + 4t$

$y = 8 + s \qquad y = 2 - 3t$

$z = 13 + 3s \qquad z = 2 - 2t$

$x = x, \; y = y, \; z = z$

$$5 + 2s = 3 + 4t \quad (A)$$
$$8 + s = 2 - 3t \quad (B)$$
$$13 + 3s = 2 - 2t \quad (C)$$

$$5 + 2s = 3 + 4t \quad (1)$$
$$8 + s = 2 - 3t \quad (2)$$
$$\boxed{s = -3t - 6} \quad (3)$$

sub into (1)

$$5 + 2(-3t - 6) = 3 + 4t$$
$$5 - 6t - 12 = 3 + 4t$$
$$-10t = 10$$
$$\boxed{t = -1}$$

sub $t = -1$ into ③

$s = -3(-1) - 6$

$\boxed{s = -3}$

sub $s = -3$ and $t = -1$ into ⓒ

$13 + 3(-3) = 2 - 2(-1)$

$13 - 9 = 2 + 2$

$\boxed{4 = 4}$ ∴ they intersect

sub $t = -1$ into ~~Ⓐ~~ Ⓑ ⓒ to parametric

find POI

$$x = 3 + 4(-1) = \boxed{-1}$$

$$y = 2 - 3(-1) = \boxed{5}$$

$$z = 2 - 2(-1) = \boxed{4}$$

$\therefore$ POI is $(-1, 5, 4)$.

Ex. 3: Find the distance between the skew lines

$L_1: (2,-1,7) + s[2,5,10]$

$L_2: (4,-21,6) + t[2,-22,17]$

Solution:

The shortest distance between skew lines is the normal.

$$\hat{n} = \overline{s} \times \overline{t}$$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 2 & 5 & 10 & 2 & 5 & 10 \\ 2 & -22 & 17 & 2 & -22 & 17 \end{array}$$

$$85i + 20j - 44k$$

$$+220i - 34j - 10k$$

$$\hat{n} = [305, -14, -54]$$

Choose a point from each line

$$\vec{P} = (2, -1, 7) \text{ to } (4, -21, 6)$$

$$= [4-2, -21-(-1), 6-7]$$

$$\vec{P} = [2, -20, -1]$$

you cannot reduce $\vec{P}$ because it is used for distance

$$\left|\operatorname{proj}_{\hat{n}} \vec{P}\right| = \left|\frac{\vec{P} \circ \hat{n}}{\hat{n} \circ \cancel{\hat{n}}}\right| \cancel{\hat{n}}$$

$$= \frac{\left|\vec{P} \circ \hat{n}\right|}{\left|\hat{n}\right|}$$

$$= \frac{\left|2(305) + (-20)(-14) + (-1)(-54)\right|}{\sqrt{305^2 + (-14)^2 + (-54)^2}}$$

$$= \frac{|610 + 280 + 54|}{\sqrt{96137}}$$

$$= \frac{944}{\sqrt{96137}}$$

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Intersection of a Line and A Plane

Here we have three cases:

1. The line and a plane intersect at a point (one solution)
2. The line is parallel to the plane (no solutions)
3. The line is on the plane (∞ solutions)

Ex. 4: Determine if the line and the plane intersect and if so find the solution.

$L_1: (-2,4,13) + m[1,2,-3]$

$\pi: (2,1,-1) + s[3,-1,-1] + t[1,4,2]$

Solution: Convert to the parametric equation of the line and scalar equation of the plane.

Line

$x = -2 + m$

$y = 4 + 2m$

$z = 13 - 3m$

For scalar equation,

$Ax + By + Cz + D = 0$,

$\hat{n} = [A, B, C]$

$$\begin{array}{cccccc} i & j & k & i & j & k \\ 3 & -1 & -1 & 3 & -1 & -1 \\ 1 & 4 & 2 & 1 & 4 & 2 \end{array}$$

$$-2i - j + 12k + k - 6j + 4i$$

$$\hat{n} = [2, -7, 13]$$

$$2x - 7y + 13z + D = 0$$

choose $(2, 1, -1) = (x, y, z)$
and solve for D

$$2(2) - 7(1) + 13(-1) + D = 0$$

$$4 - 7 - 13 + D = 0$$

$$\boxed{D = 16}$$

$$\boxed{2x - 7y + 13z + 16 = 0}$$

sub parametric line into
scalar plane

$$2(-2+m)-7(4+2m)$$
$$+13(13-3m)+16=0$$

$$-4+2m-28-14m+169-39m+16=0$$

$$-51m+153=0$$

$$\boxed{m=3}$$

sub into parametric!

$$x=-2+(3)=\boxed{1}$$
$$y=4+2(3)=\boxed{10}$$
$$z=13-3(3)=\boxed{4}$$

$\therefore$ POI is $(1, 10, 4)$.

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